Abstract Refinement Types

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Vanilla Types

12 :: **Int**

Refinement Types

$$12 :: \{v{:}\ \mathbf{Int} \mid v > 10\}$$

Refinement Types

$$12 :: \{v : \mathbf{Int} \mid v = 12\}$$

Refinement Types

$$v = 12 \Rightarrow 0 < v < 20 \wedge \text{even } v$$

$$12 :: \{v\text{: } \mathbf{Int} \mid v = 12\}$$

Refinement Types

$$v = 12 \Rightarrow 0 < v < 20 \land \text{even } v$$

$$12 :: \{v\text{: }\mathbf{Int} \mid v = 12\}$$

$$<: \{v\text{: }\mathbf{Int} \mid 0 < v < 20 \land \text{even } v\}$$

Refinement Types

$12 :: \{v : \mathbf{Int} \mid v = 12\}$

$12 :: \{v : \mathbf{Int} \mid 0 < v < 20 \land \text{even } v\}$

A max function

```haskell
max :: Int -> Int -> Int

max x y = if x > y then x else y
```

A refined max function

```
max::x:Int-> y:Int -> {v:Int | v ≥ x ∧ v ≥ y}
max x y = if x > y then x else y
```

Using max

```
max::x:Int-> y:Int -> {v:Int | v ≥ x ∧ v ≥ y}
max x y = if x > y then x else y
```

```
b = max 8 12                  -- assert (b > 0)
```

max 8 12 :: { v : **Int** | v ≥ x ∧ v ≥ y }[8/x][12/y]

Using max

```
max::x:Int-> y:Int -> {v:Int | v ≥ x ∧ v ≥ y}
max x y = if x > y then x else y
```

```
b = max 8 12                    -- assert (b > 0)
```

max 8 12 :: { v : **Int** | v ≥ 8 ∧ v ≥ 12 }

Using max

```
max::x:Int-> y:Int -> {v:Int | v ≥ x ∧ v ≥ y}
max x y = if x > y then x else y
```

```
b = max 8 12                -- assert (b > 0)
```

max 8 12 :: { v : **Int** | v ≥ 12 }

Using max

```
max::x:Int-> y:Int -> {v:Int | v ≥ x ∧ v ≥ y}
max x y = if x > y then x else y
```

```
b = max 8 12              -- assert (b > 0)
```

$v \geq 12 \Rightarrow v > 0$

max 8 12 :: $\{ v : \mathbf{Int} \mid v \geq 12 \}$ <: $\{ v : \mathbf{Int} \mid v > 0 \}$

Using max

```
max::x:Int-> y:Int -> {v:Int | v ≥ x ∧ v ≥ y}
max x y = if x > y then x else y
```

```
b = max 8 12                  -- assert (b > 0)
```

max 8 12 :: { v : **Int** | v > 0 }

Using max

```
max::x:Int-> y:Int -> {v:Int | v ≥ x ∧ v ≥ y}
max x y = if x > y then x else y
```

```
b = max 8 12                  -- assert (b > 0)✓
```

max 8 12 :: { v : **Int** | v > 0 }

Using max

```
max::x:Int-> y:Int -> {v:Int | v ≥ x ∧ v ≥ y}
max x y = if x > y then x else y
```

```
b = max 8 12            -- assert (b > 0)✓
c = max 3  5            -- assert (odd c)
```

We **get**

max 3 5 :: { v : **Int** | v ≥ 5 }

We **want**

max 3 5 :: { v : **Int** | v ≥ 5 ∧ **odd** v }

Using max

Problem:

Information of Input Refinements is Lost

We **get**

max 3 5 :: { v : **Int** | v ≥ 5 }

We **want**

max 3 5 :: { v : **Int** | v ≥ 5 ∧ **odd** v }

Our Solution

Problem:

Information of Input Refinements is Lost

Solution:

Parameterize Type Over Input Refinement

Abstract Refinements

```
max::forall <p::Int -> Prop>.
          Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

Solution:

Parameterize Type Over Input Refinement

Abstract Refinements

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

Abstract refinement

“if **both** arguments satisfy **p**,
then the **result** satisfies **p**”

Abstract Refinements

Abstract refinement

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
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"if **both** arguments satisfy **p**,
then the **result** satisfies **p**"

Abstract Refinements

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max::forall <p::Int -> Prop>.
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Abstract refinement

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Abstract Refinements

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max::forall <p::Int -> Prop>.
          Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
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Abstract refinement

“if **both** arguments satisfy **p**,
then the **result** satisfies **p**”

Using max

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd]  3  5 -- assert (odd c)
```

max :: forall <p :: **Int** -> **Prop**>.

Int<p> -> **Int**<p> -> **Int**<p>

Using max

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd]  3  5 -- assert (odd c)
```

max :: forall <p :: **Int** -> **Prop**>.

Int<p> -> **Int**<p> -> **Int**<p>

Using max

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd]  3  5 -- assert (odd c)
```

max [odd] ::

Int<p> -> **Int**<p> -> **Int**<p> [odd/p]

Using max

```
max::forall <p::Int -> Prop>.
              Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd]  3  5 -- assert (odd c)
```

max [odd] ::

{v:Int | odd v} -> {v:Int | odd v } -> {v:Int | odd v}

Using max

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd]  3  5 -- assert (odd c)
```

3 :: { v:Int | odd v }

max [odd] ::

{v:Int | odd v} -> {v:Int | odd v } -> {v:Int | odd v}

Using max

```
max::forall <p::Int -> Prop>.
          Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd] 3 5 -- assert (odd c)
```

3 :: { v:**Int** | odd v }

max [odd] 3 ::

{v:**Int** | odd v } -> {v:**Int** | odd v}

Using max

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd]  3  5 -- assert (odd c)
```

5 :: { v:Int | odd v }

max [odd] 3 ::

{v:Int | odd v } -> {v:Int | odd v}

Using max

```
max::forall <p::Int -> Prop>.
              Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd]  3  5 -- assert (odd c)
```

5 :: { v:**Int** | odd v }

max [odd] 3 5 ::

{v:**Int** | odd v}

Using max

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

```
b = max [(>0)] 8 12 -- assert (b > 0)✓
c = max [odd]  3  5 -- assert (odd c)✓
```

max [odd] 3 5 ::

{v:**Int** | odd v}

Abstract Refinements

Abstract refinement

```
max::forall <p::Int -> Prop>.
            Int<p> -> Int<p> -> Int<p>

max x y = if x > y then x else y
```

“if **both** arguments satisfy **p**, then the **result** satisfies **p**”

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A polymorphic max function

```
max :: a -> a -> a
max x y = if x > y then x else y
```

```
c = max 3 5          -- assert (odd c) ✓
```

We **instantiate**

```
a := { v:Int | odd v }
```

We **get**

```
max [{v:Int | odd v}] 3 5 ::{v:Int | odd v}
```

Type Class Constraints

```
max :: Ord a => a -> a -> a
max x y = if x > y then x else y
```

```
c = max 3 5            -- assert (odd c) ✓
```

We **instantiate**

```
a := { v:Int | odd v }
```

We **get**

```
max [{v:Int | odd v}] 3 5 ::{v:Int | odd v}
```

Unsound Reasoning

```
minus :: Num a => a -> a -> a
minus x y = x - y
```

```
b = max   3 5        -- assert (odd b)✓
c = minus 3 5        -- assert (odd c)
```

We **instantiate**

```
a := { v:Int | odd v }
```

We **get**

```
minus [{v:Int|odd v}] 3 5 :: {v:Int | odd v }
```

Abstract Refinements and Type Classes

```
max ::forall <p::a -> Prop>.
          Ord a => a<p> -> a<p> -> a<p>
max x y = if x > y then x else y
```

```
b = max [Int] [odd] 3 5-- assert (odd b)✓
```

We **get**

```
max [Int] [odd] 3 5 :: {v:Int | odd v}
```

Abstract Refinements and Type Classes

```
minus :: Num a => a -> a -> a
minus x y = x - y
```

```
b = max [Int] [odd] 3 5  -- assert (odd b)✓
c = minus [Int] 3 5      -- assert (odd b)✗
```

We **get**

```
minus [Int] 3 5 :: Int
```

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A loop function

```
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
 where go i acc | i < n = go (i+1) (f i acc)
                | otherwise = acc
```

A loop function

```
loop :: (Int -> a -> a) -> I          a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

next acc

loop iteration

final result

A loop function

initial index

initial acc

```
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

``loop f n z = $f^n(z)$''

A loop function

```
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

``loop f n z = $f^n(z)$''

A loop function

```haskell
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
 where go i acc | i < n = go (i+1) (f i acc)
                | otherwise = acc
```

```haskell
incr :: Int -> Int -> Int
incr n z = loop f n z
  where f i acc = acc + 1
```

A loop function

```haskell
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
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                 | otherwise = acc
```

```haskell
incr :: Int -> Int -> Int
incr n z = loop f n z
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```

incr acc by 1

A loop function

```haskell
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

```haskell
incr :: Int -> Int -> Int
incr n z = loop f n z
  where f i acc = acc + 1
```

incr acc by 1

Question: Does ``**incr** n z = n+z`` hold?

Answer: Proof by Induction

Inductive Proof

```haskell
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

Loop Invariant: R :: (**Int**, a)

loop iteration

accumulator

Inductive Proof

```
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

Loop Invariant: R :: (**Int**, a)

Base: R(0, z)

Inductive Step: R(i, acc) ⇒ R(i+1, f i acc)

Conclusion: R(n, loop f n z)

Induction via Abstract Refinements

```haskell
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

$$\frac{\begin{array}{l} R :: (\mathbf{Int}, a) \\ R(0, z) \\ R(i, acc) \Rightarrow \\ \quad R(i+1, f\ i\ acc) \end{array}}{R(n, \text{loop } f\ n\ z)}$$

Induction via Abstract Refinements

```
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

R :: (**Int**, a)

`r :: Int -> a -> Prop`

R(0, z)

`z :: a<r 0>`

R(i, acc) ⇒
    R(i+1, f i acc)

R(n, loop f n z)

Induction via Abstract Refinements

```
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n     = go (i+1) (f i acc)
                 | otherwise = acc
```

R :: (**Int**, a)

```
r :: Int -> a -> Prop
```

$R(0, z)$

```
z :: a<r 0>
```

$R(i, acc) \Rightarrow$
$R(i+1, f\ i\ acc)$

```
f :: i:Int -> a<r i>
       -> a<r (i+1)>
```

$$\frac{}{R(n, \text{loop}\ f\ n\ z)}$$

Induction via Abstract Refinements

```haskell
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

R :: (**Int**, a)	`r :: Int -> a -> Prop`
R(0, z)	`z :: a<r 0>`
R(i, acc) ⇒ R(i+1, f i acc)	`f :: i:Int -> a<r i> -> a<r (i+1)>`
R(n, loop f n z)	`loop f n z :: a<r n>`

Induction via Abstract Refinements

```
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

```
r :: Int -> a -> Prop

z :: a<r 0>

f :: i:Int -> a<r i>
   -> a<r (i+1)>
--------------------------------
loop f n z :: a<r n>
```

Induction via Abstract Refinements

```
loop :: (Int -> a -> a) -> Int -> a -> a
loop f n z = go 0 z
  where go i acc | i < n = go (i+1) (f i acc)
                 | otherwise = acc
```

```
loop
  :: forall <r :: Int -> a -> Prop>.
     f:(i:Int -> a<r i> -> a<r (i+1)>)
  -> n:{ v:Int | v>=0 }
  -> z:a<r 0>
  -> a<r n>
```

Induction via Abstract Refinements

```
incr :: Int -> Int -> Int
incr n z = loop f n z
  where f i acc = acc + 1
```

incr acc by 1

$$R(i, acc) \Leftrightarrow \boxed{acc = i + z}$$

```
loop
  :: forall <r :: Int -> a -> Prop>.
     f:(i:Int -> a<r i> -> a<r (i+1)>)
  -> n:{ v:Int | v>=0 }
  -> z:a<r 0>
  -> a<r n>
```

Induction via Abstract Refinements

```
incr :: Int -> Int -> Int
incr n z = loop f n z
  where f i acc = acc + 1
```

$$R(i, acc) \Leftrightarrow acc = i + z$$

```
loop [{\i acc -> acc = i + z}]
  :: f:(i:Int -> {v:a | v=i+z}
              -> {v:a | v=(i+1)+z})
  -> n:{v:Int | v>=0}
  -> z:Int
  -> {v:Int | v=n+z}
```

Induction via Abstract Refinements

```
incr :: Int -> Int -> Int
incr n z = loop f n z
  where f i acc = acc + 1
```

$$\mathcal{R}(i, acc) \Leftrightarrow acc = i + z$$

```
loop [{\i acc -> acc = i + z}]
  :: f:(i:Int -> {v:a | v=i+z}
              -> {v:a | v=(i+1)+z})
  -> n:{v:Int | v>=0}
  -> z:Int
  -> {v:Int | v=n+z}
```

Induction via Abstract Refinements

```
incr :: Int -> Int -> Int
incr n z = loop f n z
  where f i acc = acc + 1
```

$$\mathcal{R}(i, acc) \Leftrightarrow acc = i + z$$

```
loop [{\i acc -> acc = i + z}] f
  :: n:{v:Int | v>=0}
  -> z:Int
  -> {v:Int | v=n+z}
```

Induction via Abstract Refinements

```
incr :: Int -> Int -> Int
incr n z = loop f n z
  where f i acc = acc + 1
```

$$\mathcal{R}(i, acc) \Leftrightarrow acc = i + z$$

```
loop [{\i acc -> acc = i + z}] f
   :: n:{v:Int | v>=0}
   -> z:Int
   -> {v:Int | v=n+z}
```

Induction via Abstract Refinements

```haskell
incr :: Int -> Int -> Int
incr n z = loop f n z
  where f i acc = acc + 1
```

$$R(i, acc) \Leftrightarrow acc = i + z$$

```haskell
incr
  :: n:{v:Int | v>=0}
  -> z:Int
  -> {v:Int | v=n+z}
```

Induction via Abstract Refinements

```
incr :: n:{v:Int | v>=0}
     -> z:Int
     -> {v:Int | v=n+z}
incr n z = loop f n z
  where f i acc = acc + 1
```

Question: Does ``**incr** n z = n+z`` hold?

Answer: Yes

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A Vector Data Type

```
data Vec a
  = V {f :: i:Int -> a}
```

Goal: Encode the domain of Vector

Encoding the Domain of a Vector

Abstract refinement

```
data Vec <d::Int -> Prop> a
  = V {f :: i:Int<d> -> a}
```

index satisfies d

Encoding the Domain of a Vector

```
data Vec <d::Int -> Prop> a
  = V {f :: i:Int<d> -> a}
```

Encoding the Domain of a Vector

```
data Vec <d::Int -> Prop> a
  = V {f :: i:Int<d> -> a}
```

"vector defined on **positive integers**"

Vec <{\v -> v > 0}> a

Encoding the Domain of a Vector

```
data Vec <d::Int -> Prop> a
  = V {f :: i:Int<d> -> a}
```

"vector defined **only on 1**"

Vec <{\v -> v = 1}> a

Encoding the Domain of a Vector

```
data Vec <d::Int -> Prop> a
  = V {f :: i:Int<d> -> a}
```

“vector defined on **the range 0 .. n**”

Vec <{\v -> 0 ≤ v < n}> a

Encoding Domain and Range of a Vector

Abstract refinement

```
data Vec <d::Int -> Prop, r::Int -> a -> Prop> a
   = V {f :: i:Int<d> -> a<r i>}
```

value satisfies r at i

Encoding Domain and Range of a Vector

```
data Vec <d::Int -> Prop, r::Int -> a -> Prop> a
  = V {f :: i:Int<d> -> a<r i>}
```

Encoding Domain and Range of a Vector

```
data Vec <d::Int -> Prop, r::Int -> a -> Prop> a
   = V {f :: i:Int<d> -> a<r i>}
```

"vector defined on **positive integers**, with **values equal** to their **index**"

Vec <{\v -> v > 0}, {\i v -> i = v}> **Int**

Encoding Domain and Range of a Vector

```
data Vec <d::Int -> Prop, r::Int -> a -> Prop> a
   = V {f :: i:Int<d> -> a<r i>}
```

"vector defined **only on 1**,
with **values equal** to **12**"

Vec <{\v -> v = 1}, {\i v -> v = 12}> **Int**

Null Terminating Strings

```
data Vec <d::Int -> Prop, r::Int -> a -> Prop> a
   = V {f :: i:Int<d> -> a<r i>}
```

"vector defined on **the range 0 .. n,**
with its **last value equal to `\0`**"

Vec <{\v -> 0 ≤ v < n},
{\i v -> i = n-1 => v = `\0`}> **Char**

Fibonacci Memoization

```
data Vec <d::Int -> Prop, r::Int -> a -> Prop> a
   = V {f :: i:Int<d> -> a<r i>}
```

“vector defined on **positives**,
with i-th value equal to **zero or i-th fibonacci**”

Vec <{\v -> 0 ≤ v},
{\i v -> v != 0 => v = fib(i)}> **Int**

Using Vectors

- **Abstract** over d and r in vector op (get, set, …)
- **Specify** vector properties (NullTerm, FibV, …)
- **Verify** that user functions preserve properties

Using Vectors

```
type NullTerm n =
   Vec <{\v -> 0<=v<n},
         {\i v -> i=n-1 => v='\0'}> Char

upperCase
   :: n:{v: Int| v>0}
   -> NullTerm n
   -> NullTerm n
upperCase n s = ucs 0 s where

ucs i s =
   case get i s of
   '\0' -> s
   c    -> ucs (i + 1) (set i (toUpper c) s)
```

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List Data Type

```
data List a
   = N
   | C (h :: a) (tl :: List a)
```

Goal: Relate tail elements with the head

Recursive Refinements

Abstract refinement

```
data List a <p :: a -> a -> Prop>
    = N
    | C (h :: a) (tl :: List <p> (a<p h>))
```

tail elements satisfy p at h

Unfolding Recursive Refinements

```
data List a <p :: a -> a -> Prop>
  = N
  | C (h :: a) (tl :: List <p> (a<p h>))
```

Unfolding Recursive Refinements

```
data List a <p :: a -> a -> Prop>
    = N
    | C (h :: a) (tl :: List <p> (a<p h>))
```

h_1 \`C\` h_2 \`C\` h_3 \`C\` N :: List <p> a

Unfolding Recursive Refinements (1/3)

```
data List a <p :: a -> a -> Prop>
    = N
    | C (h :: a) (tl :: List <p> (a<p h>))
```

h_1 \`C\` h_2 \`C\` h_3 \`C\` N :: List <p> a

h_1 :: a

tl_1 :: List <p> (a<p h_1>)

Unfolding Recursive Refinements (2/3)

```
data List a <p :: a -> a -> Prop>
   = N
   | C (h :: a) (tl :: List <p> (a<p h>))
```

h_1 \`C\` h_2 \`C\` h_3 \`C\` N :: List <p> a

h_1 :: a

h_2 :: a<p h_1>

tl_2 :: List <p> (a<p $h_1 \wedge$ p h_2>)

Unfolding Recursive Refinements (3/3)

```
data List a <p :: a -> a -> Prop>
   = N
   | C (h :: a) (tl :: List <p> (a<p h>))
```

h_1 \`C\` h_2 \`C\` h_3 \`C\` N :: List <p> a

h_1 :: a

h_2 :: a<p h_1>

h_3 :: a<p h_1 ∧ p h_2>

N :: List <p> (a<p h_1 ∧ p h_2 ∧ p h_3>)

Increasing Lists

```
data List a <p :: a -> a -> Prop>
   = N
   | C (h :: a) (tl :: List <p> (a<p h>))
```

```
type IncrL a = List <{\hd v -> hd ≤ v}> a
```

h_1 \`C\` h_2 \`C\` h_3 \`C\` N :: IncrL a

Increasing Lists

```
data List a <p :: a -> a -> Prop>
  = N
  | C (h :: a) (tl :: List <p> (a<p h>))
```

```
type IncrL a = List <{\hd v -> hd ≤ v}> a
```

h_1 \`C\` h_2 \`C\` h_3 \`C\` N :: IncrL a

h_1 :: a

h_2 :: { v:a | $h_1 \leq v$ }

h_3 :: { v:a | $h_1 \leq v \wedge h_2 \leq v$ }

N :: IncrL { v:a | $h_1 \leq v \wedge h_2 \leq v \wedge h_3 \leq v$ }

Increasing Lists

```
data List a <p :: a -> a -> Prop>
   = N
   | C (h :: a) (tl :: List <p> (a<p h>))
```

```
type IncrL a = List <{\hd v -> hd ≤ v}> a
```

Demo from
http://goto.ucsd.edu/~rjhala/liquid/haskell/blog/

Outline

Introduction

- Monomorphic Refinements

Applications

- Refinements and Type Classes
- Inductive Refinements
- Indexed Refinements
- **Recursive Refinements**

Evaluation

Outline

Introduction

- Monomorphic Refinements

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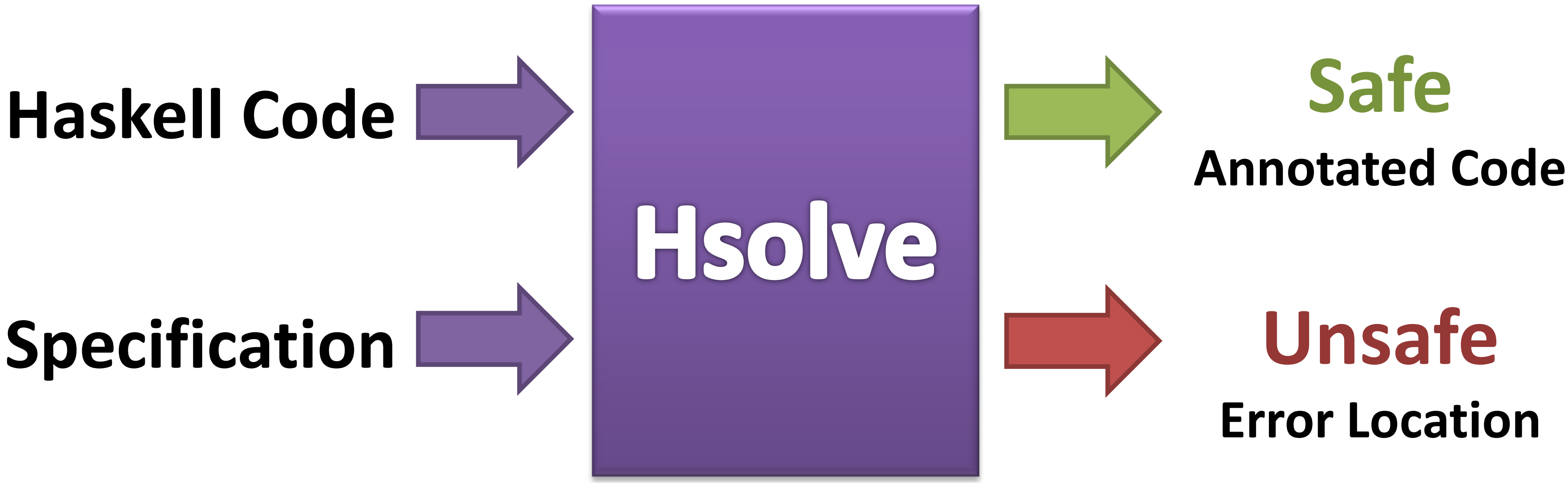
- Recursive Refinements

Evaluation

Our Tool


Haskell Code
Specification
Hsolve
Safe
Annotated Code
Unsafe
Error Location

Implementing HSolve

Hsolve = Liquid Types [PLDI 2008]
+ Abstract Refinements

Refinement Abstraction

Does not increase complexity

Refinement Application

Refinement application is **inferred**

Benchmarks

Program	LOC	Annotations	Time (s)
Micro	32	23	2
Vector	33	53	5
ListSort	29	5	3
Data.List.sort	71	4	8
Data.Set.Splay	136	24	13
Data.Map.Base	1395	152	136
Total	1696	261	167

Benchmarks

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Total	1696	261	167

Data.Map.Base

```
data Map k a
  = Bin Size k a (Map k a) (Map k a)
  | Tip
```

``Relate keys of **left** and **right** subtrees with the **root** key``

Data.Map.Base

```
data Map k a < l :: root:k -> k -> Prop
             , r :: root:k -> k -> Prop>
  = Bin (sz :: Size) (key :: k) (value :: a)
        (left  :: Map <l, r> (k <l key>) a)
        (right :: Map <l, r> (k <r key>) a)
  | Tip
```

``Relate keys of **left** and **right** subtrees with the **root** key``

Data.Map.Base

```
type OMap k a =
  Map <{\root v -> v<root}
      ,{\root v -> v>root}> k a
```

OMap is a BST

``left : keys less than the root

right: keys greater than the root``

Data.Map.Base

```
{-@ balanceL :: kcut:k -> a -> OMap {v:k | v < kcut} a -> OMap {v:k| v > kcut} a -> OMap k a @-}
balanceL :: k -> a -> Map k a -> Map k a -> Map k a

balanceL k x l r = case r of
  Tip -> case l of
        Tip -> Bin 1 k x Tip Tip
        (Bin _ _ _ Tip Tip) -> Bin 2 k x l Tip
        (Bin _ lk lx Tip (Bin _ lrk lrx _ _)) -> Bin 3 lrk lrx (Bin 1 lk lx Tip Tip) (Bin 1 k x Tip Tip)
        (Bin _ lk lx ll@(Bin _ _ _ _ _) Tip) -> Bin 3 lk lx ll (Bin 1 k x Tip Tip)
        (Bin ls lk lx ll@(Bin lls _ _ _ _) lr@(Bin lrs lrk lrx lrl lrr))
          | lrs < ratio*lls -> Bin (1+ls) lk lx ll (Bin (1+lrs) k x lr Tip)
          | otherwise -> Bin (1+ls) lrk lrx (Bin (1+lls+size lrl) lk lx ll lrl) (Bin (1+size lrr) k x lrr Tip)
  (Bin rs _ _ _ _) -> case l of
        Tip -> Bin (1+rs) k x Tip r
        (Bin ls lk lx ll lr)
          | ls > delta*rs -> case (ll, lr) of
              (Bin lls _ _ _ _, Bin lrs lrk lrx lrl lrr)
                | lrs < ratio*lls -> Bin (1+ls+rs) lk lx ll (Bin (1+rs+lrs) k x lr r)
                | otherwise -> Bin (1+ls+rs) lrk lrx (Bin (1+lls+size lrl) lk lx ll lrl) (Bin (1+rs+size lrr) k x lrr r)
              (_, _) -> error "Failure in Data.Map.balanceL"
          | otherwise -> Bin (1+ls+rs) k x l r
```

Data.Map.Base

```
{-@ balanceL :: kcut:k -> a -> OMap {v:k | v < kcut} a -> OMap {v:k| v > kcut} a -> OMap k a @-}
balanceL :: k -> a -> Map k a -> Map k a -> Map k a

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         (Bin _ lk lx ll@(Bin _ _ _ _ _) Tip) -> Bin 3 lk lx ll (Bin 1 k x Tip Tip)
         (Bin ls lk lx ll@(Bin lls _ _ _ _) lr@(Bin lrs lrk lrx lrl lrr))
           | lrs < ratio*lls -> Bin (1+ls) lk lx ll (Bin (1+lrs) k x lr Tip)
           | otherwise -> Bin (1+ls) lrk lrx (Bin (1+lls+size lrl) lk lx ll lrl) (Bin (1+size lrr) k x lrr Tip)
  (Bin rs _ _ _ _) -> case l of
         Tip -> Bin (1+rs) k x Tip r
         (Bin ls lk lx ll lr)
            | ls > delta*rs -> case (ll, lr) of
                (Bin lls _ _ _ _, Bin lrs lrk lrx lrl lrr)
                  | lrs < ratio*lls -> Bin (1+ls+rs) lk lx ll (Bin (1+rs+lrs) k x lr r)
                  | otherwise -> Bin (1+ls+rs) lrk lrx (Bin (1+lls+size lrl) lk lx ll lrl) (Bin (1+rs+size lrr) k x lrr r)
                (_, _) -> error "Failure in Data.Map.balanceL"
            | otherwise -> Bin (1+ls+rs) k x l r
```

Conclusion

Abstract Refinements

- Increase expressiveness without complexity
- Behave as uninterpreted functions in logic
- Relate arguments with result, i.e., max
- Relate expressions inside a structure, i.e., Vec, List
- Express recursive properties, i.e., List
- Express inductive properties, i.e., loop

Thank you!